and optimal strategies (x_1, x_2) *and* (y_1, y_2) *are determined by*

$$\frac{x_1}{x_2} = \frac{v_{22} - v_{21}}{v_{11} - v_{12}}, \quad \frac{y_1}{y_2} = \frac{v_{22} - v_{12}}{v_{11} - v_{21}}.$$

and the value (v) *of the game to the player A is given by*

$$v = \frac{v_{11}v_{22} - v_{12}v_{21}}{v_{11} + v_{22} - (v_{12} + v_{21})}.$$

[Meerut 2002; Rohilkhand 92]

Proof. Let a mixed strategy for player A be given by (x_1, x_2) where $x_1 + x_2 = 1$. Thus if player B moves his first strategy, the net expected gain of A will be $E_1(x) = v_{11}x_1 + v_{21}x_2$;
and if B moves his second strategy, the net expected gain of A will be $E_2(x) = v_{12}x_1 + v_{22}x_2$.

But, player A wants to maximize his minimum expected gain. So the value of the game (v) must be minimum of $E_1(x)$ and $E_2(x)$, *i.e.* $E_1(x) \geq v$, $E_2(x) \geq v$.

Thus for the player A, we have to find $x_1 \geq 0$, $x_2 \geq 0$, and v to satisfy the following three relationships (as obtained in ***Sec.19·10***) :

$$v_{11}x_1 + v_{21}x_2 \geq v, \quad \text{...(19·33)}$$

$$v_{12}x_1 + v_{22}x_2 \geq v, \quad \text{...(19·34)}$$

$$x_1 + x_2 = 1. \quad \text{...(19·35)}$$

For optimum strategies, inequalities (19·33) and (19·34) become strict equations, ***i.e.***

$$v_{11}x_1 + v_{21}x_2 = v, \quad \text{...(19·36)}$$

$$v_{12}x_1 + v_{22}x_2 = v. \quad \text{...(19·37)}$$

Subtracting equation (19·37) from the equation (19·36), we get

$$(v_{11} - v_{12})\, x_1 + (v_{21} - v_{22})\, x_2 = 0. \quad \text{...(19·38)}$$

which gives

$$\frac{x_1}{x_2} = \frac{v_{22} - v_{21}}{v_{11} - v_{12}}. \quad \text{...(19·39)}$$

Hence, we evaluate x_1 and x_2 separately by using the equation (19·35),

$$x_1 = \frac{v_{22} - v_{21}}{v_{11} + v_{22} - (v_{12} + v_{21})} \quad \text{...(19·40)}$$

$$x_2 = 1 - x_1 = \frac{v_{11} - v_{12}}{(v_{11} + v_{22}) - (v_{12} + v_{21})}. \quad \text{...(19·41)}$$

The value of the game can be obtained by substituting the values of x_1 and x_2 in either of the equations (19·36) and (19·37) to obtain

$$v = \frac{v_{11}(v_{22} - v_{21})}{v_{11} + v_{22} - (v_{12} + v_{21})} + \frac{v_{21}(v_{11} - v_{12})}{v_{11} + v_{22} - (v_{12} + v_{21})} \text{ or } v = \frac{v_{11}v_{22} - v_{21}v_{12}}{v_{11} + v_{22} - (v_{12} + v_{21})}. \quad \text{...(19·42)}$$

In the same manner for the player B, find $y_1 \geq 0$, $y_2 \geq 0$, and v to satisfy the following three relations :

$$v_{11}y_1 + v_{12}y_2 \leq v, \quad \text{...(19·43)}$$

$$v_{21}y_1 + v_{22}y_2 \leq v, \quad \text{...(19·44)}$$

$$y_1 + y_2 = 1. \quad \text{...(19·45)}$$

Here it should be remembered that the player B wants to minimize his maximum loss.

Again for optimum strategies of player B, consider the inequalities (19·43) and (19·44) as strict equations and obtain

$$\frac{y_1}{y_2} = \frac{v_{22} - v_{12}}{v_{11} - v_{21}}. \quad \text{...(19·46)}$$

Using the equation (19·45)

$$y_1 = \frac{v_{22} - v_{12}}{v_{11} + v_{22} - (v_{21} + v_{12})} \quad \text{...(19·47)}$$

$$y_2 = 1 - y_1 = \frac{v_{11} - v_{21}}{v_{11} + v_{22} - (v_{21} + v_{12})}. \quad \text{...(19·48)}$$

Substituting values of y_1 and y_2 in either of the equation (19·43) or (19·44), to obtain the value

$$v = \frac{v_{11}v_{22} - v_{21}v_{12}}{v_{11} + v_{22} - (v_{12} + v_{21})}, \quad ...(19·49)$$

which is the same as desired by the *minimax theorem.*

If ratios x_1/x_2 and y_1/y_2 are both positive, these will give acceptable values of x_1, x_2, y_1 and y_2. A solution satisfying all constraints including non-negativity, may be obtained.

This proves the requied results.

Further for such games, in a payoff matrix *the largest and second largest elements must lie on one of the diagonals.* This implies that there are only 8 possible orderings (instead of 24) of entries v_{11}, v_{12}, v_{21}, v_{22} without saddle point.

These possibilities are :

$$\begin{Bmatrix} v_{11} \ge v_{22} \ge v_{12} \ge v_{21} \\ v_{11} \ge v_{22} \ge v_{21} \ge v_{12} \end{Bmatrix}, \begin{Bmatrix} v_{22} \ge v_{11} \ge v_{21} \ge v_{12} \\ v_{22} \ge v_{11} \ge v_{12} \ge v_{21} \end{Bmatrix}, \begin{Bmatrix} v_{12} \ge v_{21} \ge v_{11} \ge v_{22} \\ v_{12} \ge v_{21} \ge v_{22} \ge v_{11} \end{Bmatrix}, \begin{Bmatrix} v_{21} \ge v_{12} \ge v_{11} \ge v_{22} \\ v_{21} \ge v_{12} \ge v_{22} \ge v_{11} \end{Bmatrix},$$

It can be easily verified that, with all above orderings, ratios x_1/x_2 and y_1/y_2 are non-negative.

Remark. If these formulae for x_1, x_2, y_1, y_2 and v are applied to a 2×2 games with saddle point, these may give an incorrect solution.

Q. 1. For the game $\begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$, where a, b, c, d are all non-negative ≥ 0, prove that the optimal strategies are :

$$A : \left(\frac{c+d}{a+b+c+d}, \frac{a+b}{a+b+c+d} \right), B : \left(\frac{b+d}{a+b+c+d}, \frac{a+c}{a+b+c+d} \right) \text{ and } v = \frac{ad - bc}{a+b+c+d}$$

[Meerut M.Sc (Math.) 96]

2. Given the 2×2 payoff matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Suppose player A adopts the strategy (x, y); while B adopts the strategy (u, v) where x, y, u, v are all ≥ 0, such that $x + y = u + v = 1$.
(i) Express A's expected gain z in terms of x, y, u, v and a, b, c, d.
(ii) What is the effect on z of adding the same constant k to each element of the payoff matrix ?
(iii) What is the effect on z of multiplying each element of payoff matrix by the same constant k ?
(iv) How are the optimal strategies affected by these operations on payoff matrix.

3. If $G = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a non-strictly determined matrix game, then show that either
(i) $a < b, a < c, d < c, d < b$ or (ii) $a > b, a > c, d > c, d > b$. **[Kanpur 2000]**

4. Prove that 2×2 matrix game is strictly determined only if its principal diagonal elements are either strictly greater or strictly smaller than the other elements.

5. If all the elements of the payoff matrix of a game are non-negative and every column of this matrix has at least one positive element, then the value of the corresponding game is positive.

6. What do you mean by saddle point of a two-person zero-sum game ? In a 2×2 game if the largest and second largest elements lie along a diagonal, then prove that the game has no saddle point.

7. Let (a_{ij}) be the payoff matrix for a two-person zero-sum game. Examine the game for saddle point under the following orderings of its elements :
(a) $a_{21} \ge a_{22} \ge a_{11} \ge a_{12}$ (b) $a_{11} \le a_{12} \le a_{21} \le a_{22}$ (iii) $a_{12} \le a_{22} \le a_{11} \le a_{21}$ (iv) $a_{22} \le a_{11} \le a_{12} \le a_{21}$

8. For a two-person zero-sum game, the payoff matrix for player A is $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ with no saddle point. Obtain the optimal strategies (x_1, x_2) and (y_1, y_2) respectively.

Note. Students are advised to solve 2×2 games without saddle point by originally constructing the relationship for both the players instead of using the formulae (19·40), (19·41), (19·47), (19·48) and (19·49) directly.

19·13-1. Arithmetic Method for (2 × 2) Games

Arithmetic method provides an easy technique for obtaining the optimum strategies for each player in (2 × 2) games without saddle point. This method consists of the following steps :

Step 1. Find the difference of two numbers in column I, and put it under the column II, neglecting the negative sign if occurs.

Step 2. Find the difference of two numbers in column II, and put it under the column I, neglecting the negative sign if occurs.

Step 3. Repeat the above two steps for the two rows also.

The values thus obtained are called the ***oddments.*** These are the frequencies with which the players must use their courses of action in their optimum strategies.

The above steps can be explained by the following example :

Example 8. *Two players A and B without showing each other, put on a table a coin, with head or tail up. A wins Rs. 8 when both the coins show head and Re. 1 when both are tails. B wins Rs. 3 when the coins do not match. Given the choice of being matching player (A) or non-matching player (B), which one would you choose and what would be your strategy ?*

Solution. The payoff matrix for A is found to be

		Player B H	Player B T		
Player A	H	8	−3	4	$\frac{4}{11+4}=\frac{4}{15}$
	T	−3	1	11	$\frac{11}{11+4}=\frac{11}{15}$

Since no saddle point is found, the optimal strategies will be the mixed strategies.

Step 1. Taking the difference of two numbers in column I, we find $8-(-3)=11$, and put it under column II.

Step 2. Taking the difference of two numbers in column II, we find $(-3-1)=-4$, and put the number 4 (neglecting the − ve sign) under column I.

Step 3. Repeat the above two steps for the two rows also.

Thus for optimum gains, player A must use strategy H with probability $4/15$ and strategy T with probability $11/15$, while player B must use strategy H with probability $4/15$ and strategy T with probabiltiy $11/15$.

Step 4. To obtain the value of the game any of the following expressions may be used.

Using *B*'s oddments :

$$B \text{ plays } H, \text{ value of the game, } v = \text{Rs. } \frac{4\times 8+11\times(-3)}{11+4} = \text{Rs. } \left(-\frac{1}{15}\right)$$

$$B \text{ plays } T, \text{ value of the game } v = \text{Rs. } \frac{4\times(-3)+11\times 1}{11+4} = \text{Rs. } \left(-\frac{1}{15}\right).$$

Using *A*'s oddments :

$$A \text{ plays } H, \text{ value of the game, } v = \text{Rs. } \frac{4\times 8+11\times(-3)}{4+11} = \text{Rs. } \left(-\frac{1}{15}\right)$$

$$A \text{ plays } T, \text{ value of the game, } v = \text{Rs. } \frac{4\times(-3)+11\times 1}{4+11} = \text{Rs. } \left(-\frac{1}{15}\right).$$

The above values of v are equal only if the sum of the oddments vertically and horizontally are equal. Cases in which it is not so will be discussed later.

Thus the complete solution of the game is : (i) optimum strategy for A is $(4/15, 11/15)$. and for B is $(4/15, 11/15)$.

(ii) value of the game to A is $v=$ Rs. $(-1/15)$ and to B is $1/15$.

Thus, player A gains Rs. $(-1/15)$, *i.e.*, he loses Rs. $(1/15)$ which B, in turn, gets.

Note. Arithmetic method is easier than the algebraic method but it cannot be applied to larger games.

EXAMINATION PROBLEMS

Solve the following 2 × 2 games without saddle points :

1. $A\overset{B}{\begin{bmatrix}5 & 1\\ 3 & 4\end{bmatrix}}$ **2.** $A\overset{B}{\begin{bmatrix}6 & -3\\ -3 & 0\end{bmatrix}}$ **3.** $A\overset{B}{\begin{bmatrix}2 & 5\\ 7 & 3\end{bmatrix}}$ **4.** $A\overset{B}{\begin{bmatrix}-4 & 6\\ 2 & -3\end{bmatrix}}$

[Ans. 1. (1/5, 4/5), (3/5, 2/5) , $v=17/5$] **2.** (1/4, 3/4) for both player, $v=-3/4$] **3.** (4/7, 3/7), (2/7, 5/7), $v=29/7$ **4.** (1/3, 2/3), (3/5, 2/5), $v=0$]

5. $A\overset{B}{\begin{bmatrix}3 & -2\\ -2 & 3\end{bmatrix}}$ **6.** $A\overset{B}{\begin{bmatrix}2 & 5\\ 4 & 1\end{bmatrix}}$ **7.** $\begin{bmatrix}4 & -4\\ -4 & 4\end{bmatrix}$

[Ans. (1/2, 1/2), (1/2, 1/2), $v=1/2$] **[Ans.** A (1/2, 1/2), B (1/2, 1/2), $v=0$]

8. Two players A and B match coins. If the coin match, then A wins one unit of value, if the coins do not match, then B wins one unit of value. Determine optimum strategies for the players and the value of the game.

[Hint. Formulation of the game is : $A\ \overset{B}{\begin{matrix} & H & T\\ H & +1 & -1\\ T & -1 & +1\end{matrix}}$ **[Ans.** (1/2, 1/2), (1/2, 1/2), $v=0$]

9. A and B each take out one or two matches and guess how many matches opponent has taken. If one of the players guess correctly then the looser has to pay him as many rupees as the sum of the numbers held by both players. Otherwise, the payout is zero. Write down the payoff matrix and obtain the optimal strategies of both players.

[**Hint.** Formulation of the game is :

$$\begin{array}{cc} & B \\ & \begin{array}{cc} 1 & 2 \end{array} \\ A \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \end{array}$$

[**Ans.** (2/3, 1/3), (2/3, 1/3), $v = 4/3$]

10. In a game of matching coins with two players, suppose A wins one unit of value when there are two heads, wins nothing when there are two tails, and loses 1/2 unit of value when there are one head and one tail. Determine the payoff matrix, the best strategies for each player, and the value of the game to A.

[**Hint.** Formulation of the game is :

$$\begin{array}{cc} & B \\ & \begin{array}{cc} H & T \end{array} \\ A \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \end{array}$$

[**Ans.** (1/4, 3/4), (1/4, 3/4), $v = -1/8$]

11. Consider a modified form of 'matching biased wins' game problem. The matching player is paid eight rupees if the two wins turn both heads and one rupee if the two wins turn both tails. The non-matching player is paid three rupees when the two wins do not match. Given the choice of being the matching or non-matching player, which one would you choose and what would be your strategy ? **[IAS (Maths. 97]**

12. Solve the following game and determine the value of the game :

		Player Y	
		Strategy 1	Strategy 2
Player X	Strategy 1	4	1
	Strategy 2	2	3

[Allahabad (M.B.A.) 98]

[**Ans.** The optimum strategies for the two players are :
S_X = (1/4, 3/4) and S_Y = (1/2, 1/2) and the value of game = 10/4.]

19·14 PRINCIPLE OF DOMINANCE TO REDUCE THE SIZE OF THE GAME

For easiness of solutions, it is always convenient to deal with smaller payoff matrices. Fortunately, the size of the payoff matrix can be considerably reduced by using the so called *principle of dominance.* Before stating this principle, let us define a few important terms.

Inferior and Superior Strategies. *Consider two n-tuples* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ *and* $\mathbf{b} = (b_1, b_2, \ldots, b_n)$. *If* $a_i \geq b_i$ *for all* $i = 1, 2, \ldots, n$, *then for player A the strategy corresponding to* **b** *is said to be* ***inferior*** *to the strategy corresponding to* **a**; *and equivalently, the strategy corresponding to* **a** *is said to* ***be superior*** *to the strategy corresponding to* **b.**

For player B, the above situation will be reversed, because player A's gain-matrix is player B's loss-matrix.

Dominance. *An n-tuple* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ *is said to* ***dominate*** *the n-tuple* $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ *if* $a_i \geq b_i$ *for all* $i = 1, 2, \ldots, n$. *The superior strategies are said to dominate the inferior ones.*

Thus a player would not like to use inferior strategies which are dominated by other's. Now we are able to state the principle of dominance as follows :

Principle of Dominance. *If one pure strategy of a player is better or superior than another one (irrespective of the strategy emplayed by his opponent), then the inferior strategy may be simply ignored by assigning a zero probability while searching for optimal strategies.*

Theorem 19.4 (Dominance Property). *Let* $\mathbf{A} = [v_{ij}]$ *be the payoff matrix of an* $m \times n$ *rectangular game. If the ith row of* **A** *is dominated by the rth row of* **A**, *then the deletion of ith row of* **A** *does not change the set of optimal strategies for the row player (player* **A**).

Further, if the jth column of **A** *dominates the kth column of* **A**, *then the deletion of jth column of* **A** *does not change the set of optimal strategies for the column player (player B).*

Proof. Given that

$$v_{ij} \leq v_{rj}, \text{ for all } j = 1, 2, \ldots, n \text{ and } v_{ij} \neq v_{rj} \text{ for at least one } j \qquad \ldots(1)$$

Let $\mathbf{y}^* = (y_1^*, y_2^*, \ldots, y_n^*)$ be an optimal strategy for the column player B. It follows from (1) that